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CONSTRUCTIBLE SETS OF LINEAR DIFFERENTIAL EQUATIONS AND EFFECTIVE RATIONAL APPROXIMATIONS OF POLYLOGARITHMIC FUNCTIONS

BY

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ABSTRACT

The aim of this paper is to investigate rational approximations to solutions of some linear Fuchsian differential equations from the perspective of moduli of linear differential equations with fixed monodromy group. One of the main arithmetic applications concerns the study of linear forms involving polylogarithmic functions. In particular, we give an explanation of the well-poised hypergeometric origin of Rivoal's construction on linear forms involving odd zeta values.

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1. Introduction

The purpose of the present paper is to investigate rational approximations to solutions of certain Fuchsian linear differential equations from the viewpoint of the theory of moduli of differential equations with a fixed monodromy group.

We shall only consider differential equations defined on the Riemann sphere, $\mathbb{P}_1(\mathbb{C})$. Given a family of analytic functions on $\mathbb{P}_1(\mathbb{C})$, the existence of a linear combination of its members with zeros of high order will be seen to follow from the study of the local behavior of Fuchsian differential equations. This, in turn, reveals a connection between Fuchsian differential equations and diophantine approximations of special Siegel G-functions. Riemann [Rie] initiated this viewpoint in his study of Gauss's continued fraction expansion of

$$(1.1) \qquad {}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix} \middle| x\right) / {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right)$$

(see also [Hu1]).

More recently, G. V. Chudnovski [Chu2] has obtained many interesting results about linear forms involving generalized hypergeometric functions. Subsequently, Y. Nesterenko [Ne1], [Ne2] carried out more precise studies of this topic.

Here, we obtain some formulae that play an important role in the theory of Padé approximation.

Following Riemann, we use the notion of a local system to solve the realization problem: given a finite subset S of $\mathbb{P}_1(\mathbb{C})$ that contains ∞ and given local numerical data of rank m on $Z =: \mathbb{P}_1(\mathbb{C})\backslash S$, we are asked to explicitly construct and analyze the behavior of Fuchsian differential equations that give rise to the given local system.

This construction comes from the choice of a basis of this local system which is "good" at every singular point. This basis is the Levelt basis [Le], [AB], a triangular basis which is related to the monodromy of this local system.

If one uses this basis, it is very easy to find the exponents of the linear Fuchsian differential equation related to this local system.

When the local system has no accessory parameters, that is, when it is "rigid" according to the terminology of Katz [Ka], the Fuchsian differential equation is unique and we obtain a Padé linear form as a special case of the contiguity relations. Here, we note that Katz [Ka] works only with irreducible differential equations, but in many special cases we extend his results to reducible equations related to polylogarithmic functions.

The rigidity of the local system or, equivalently, the differential equation, has important arithmetic applications. In particular, the uniqueness of the equation corresponding, respectively, to the generalized hypergeometric functions [Chu2] or to the polylogarithmic functions [Hu1], [Ne1], [Hu4] enables us to establish many new explicit formulae for Padé approximations. For example, since Meijer's G-functions [Gu] satisfy Fuchsian differential equations, they can be expressed in terms of approximants given by our method.

Our results provide a new viewpoint for the study of linear forms involving polylogarithmic functions. We give a uniform exposition of certain aspects of the famous works of Apery on the irrationality of $\zeta(3)$ and generalize the constructions of Ball, Rivoal and Zudilin [Ba], [Ri], [Zu1], [Zu2] on the arithmetic of linear forms in odd zeta values.

In particular, we provide a new exposition of the well-poised origin of Rivoal's construction. (See [Fi1] and [Fi2] for a recent account of the arithmetic nature of these functions.)

Admittedly, our general method does not so far directly reproduce or extend the famous arithmetic results obtained by the cited authors; our aim is to illustrate how ad hoc calculations can be replaced by a systematic procedure.

In order to make our exposition self-contained, we shall first review in the next subsections some relevant background material, omitting the proofs.

1.1 Hypergeometric power series. We begin with the following observation and definition.

Definition 1: Let $F(X) = \sum_{m=0}^{\infty} c(m)X^m$ be a formal power series.

We define the relatively prime polynomials P and Q by the relation

$$\frac{c(m+1)}{c(m)} = \frac{P(m)}{Q(m)(m+1)}$$

where P and Q are relatively prime polynomials. The series $F(X)$ formally satisfies the following differential equation:

$$Q(\theta)F = XP(\theta)F$$

where $\theta = X\frac{d}{dX}$.

We refer, for instance, to [Er] for basic definitions of classical generalized hypergeometric power series ${}_pF_q$ in a single variable x with parameters

$$a_i, b_j, \quad i = 1, \dots, p, \quad j = 1, \dots, q.$$

In this case it turns out that a factorization of the corresponding polynomials P and Q leads to the following classical definition:

$$(1.2) \qquad {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} x^n.$$

Here $(a)_n$ is a Pochhammer symbol:

$$(1.3) \qquad (a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1).$$

1.2 Fuchsian linear differential equations without accessory parameters. Consider Fuchsian linear differential equations of order m with p regular singular points

$$(1.4) \qquad S = \{a_1, a_2, \dots, a_{p-1}, \infty\}.$$

Denote by $L_{m,p}$ this set of differential equations and let $\psi(x)$ be the polynomial

$$\psi(x) = \prod_{i=1}^{p-1}(x - a_i).$$

An element E of $L_{m,p}$ can be written

$$(1.5) \qquad E: \quad y^{(m)} + \frac{\phi_1(x)}{\psi(x)} y^{(m-1)} + \cdots + \frac{\phi_m(x)}{\psi(x)^m} y = 0$$

where the polynomials $\phi_i(x)$ are of degree at most $i(p-2)$.

One can deduce that the dimension δ of this vector space is

$$(1.6) \qquad \delta = \sum_{k=1}^{m}(p-2)k + 1) = \left((p-2)\frac{m(m+1)}{2}\right) + m.$$

One can identify $L_{m,p}$ with the vector space $\mathbb{C}^\delta$.

If $V = \operatorname{Sol} L_{m,p}$ is the vector space of solutions of E and G is the monodromy group of E, then V is a G module.

Remark 1: In [Chu3] this subspace V is called a Fuchsian module. See also [Ber] and [Ber, Beu].

Let us consider the set $Sing$ of regular singular points of E and $Exp(a)$ the set of exponents of E at a.

For a regular singular point a, let us denote by $|V_a|$ the number of linear conditions on E which are necessary and sufficient for the absence of logarithmic terms from the solution of this differential equation at a. In this case $|V_a|$ is the number of holomorphic linearly independent solutions $\psi_a(x)$ at a such that $(x-a)^{e_a}\psi_a(x)$ is a solution of E belonging at the exponent e_a.

(If, at a singular point, one knows that two exponents differ by a natural number then, in general, one has a logarithmic solution.

Using a result of Ince [In] (page 404), we can find linear conditions such that all solutions relative to a particular exponent e_a at a are free from logarithms.)

We put

$$|V| = \sum_{a\in Sing(a)} |V_a|.$$

We can conclude that A is the space of moduli of a vector space $L_{p,m,V,G}$.

If V and G are given, the vector space $L_{p,m,V,G}$ depends on p, m, V, G; then its dimension is

$$|A| = \left[(p-2)\frac{m(m+1)}{2} + m\right] - [(pm-1) + |V|]. \tag{1.7}$$

The dimension $|A|$ is known as the number of accessory parameters of an element E of $L_{m,p,V,G}$.

If $p = 3$ and $m = 1$, we obtain $|V| = 0$ and $|A| = 0$, that is we have the famous Riemann P scheme [Rie]. (In general $|A| > 0$, but we do not study this case in this article.)

If $|A| = 0$, then E is without accessory parameters.

In this case, we can associate to E a $p \times m$ matrix called a "Riemann scheme" (or "Riemann symbol") R_E which gives exactly the roots of the indicial equation at the regular singular points of E.

The map between this Riemann scheme and E is one-to-one (see [Hu2]). The general case gives

$$R_E = \begin{pmatrix} \underline{a_1} & \underline{a_2} & \cdots & \underline{\infty} & \\ e_1^1 & e_1^2 & \cdots & e_1^\infty & \\ e_2^1 & e_2^2 & \cdots & e_2^\infty & |x \\ \vdots & \vdots & \vdots & & \\ e_m^1 & e_m^2 & \cdots & e_m^\infty & \end{pmatrix}. \tag{1.8}$$

In the sequel we only consider the hypergeometric case.

1.2.1 The hypergeometric case. In this case

$$p = 3; \quad |V| = (m-2)(m-1)/2.$$

Since at 1 there exists a basis of $m-1$ holomorphic solutions, we have

$$0, 1, \ldots, m-2$$

for the corresponding exponents.

Let us give a sketch of the proof of the unicity of the Fuchsian differential equation E in this case.

We consider the restriction E_1 of the operator E to the subspace V_1 generated by this basis of holomorphic solutions at 1. The operator E_1 is of order $m-1$ and for this differential equation, 1 is an ordinary regular point.

Furthermore, the fact that there does not exist at the singularity 1 any logarithmic solutions adds

$$1 + 2 + \cdots + m - 2 = |V|$$

new linear relations.

1.3 LEVELT'S CONSTRUCTION OF THE HYPERGEOMETRIC DIFFERENTIAL EQUATION. **The following theorem, which is Levelt's construction of the hypergeometric differential equation [Le], is the main tool of our paper.**

Let

$$Z' = \mathbb{P}_1(\mathbb{C}) \backslash \{0, 1, \infty\}$$

and let W denote the universal covering of Z'. Let $M(W)$ be the field of meromorphic functions on W and p the projection of W onto Z'. At each point of W, we can consider p to be a local parameter.

THEOREM 1 (Levelt): *Let V be an m-dimensional subspace over $\mathbb{C}$ of the vector space W^m satisfying:*

(HF_1) *The elements of V are holomorphic on W.*

(HF_2) *The elements of V are of finite order at $0, \infty, 1$.*

(HF_3) *If $\pi_1(Z')$ denotes the group of all covering transformations of Z', to each $\delta \in \pi_1(Z')$ there corresponds an automorphism δ^* of $M(W)$ defined by*

$$\forall w \in W, \quad \forall f \in M(W), \quad (\delta^* f)(w) = f(\delta w)$$

(i.e. δ^* is the monodromy's operator).

Then δ^* is an automorphism of V.

(HF_4) *Let*

$$\sum_{Sing=0,1,\infty} (exp)$$

be the sum of all the exponents; then

$$\sum_{Sing=0,1,\infty} (exp) = m(m-1)/2$$

(Fuchs relation).

(HF_5) *The $(m-1)$-dimensional initial value problem at 1 is solvable (i.e. there exist $m-1$ $\mathbb{C}$ linearly independent holomorphic functions of V at 1).*

(For a more precise and equivalent definition see [Le] theorems 2.8, page 381 and 2.10, page 383).

Then V is the solution space of the hypergeometric differential equation

$$(1.9)\quad ((\theta-\gamma_1)(\theta-\gamma_2)\cdots(\theta-\gamma_m) - x(\theta+\alpha_1)(\theta+\alpha_2)\cdots(\theta+\alpha_m))y(x) = 0.$$

Its Riemann scheme is

$$(1.10)\qquad R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \gamma_1 & \alpha_1 & 0 & \\ \gamma_2 & \alpha_2 & 1 & \\ \vdots & \vdots & 2 & |x \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ \gamma_m & \alpha_m & d & \end{pmatrix}$$

with

$$(1.11)\qquad \sum_{i=1}^{m}\gamma_i + \sum_{i=1}^{m}\alpha_i + d = m.$$

Let us suppose that $\gamma_1 = 0$, i.e. there exists a holomorphic solution $f(x)$ at 0 for the hypergeometric operator.

In general we put

$$\gamma_1 = 1-\beta_m = 0, \quad \gamma_2 = 1-\beta_2, \quad \ldots, \quad \gamma_{m-1} = 1-\beta_{m-1}$$

and within a multiplicative constant $f(x)$ is the hypergeometric power series of parameters

$$\alpha_1, \alpha_2, \ldots, \alpha_m; \quad \beta_1, \ldots, \beta_{m-1}, \beta_m = 1.$$

The last exponent d at 1 is very important. If, in particular, its value is a nonnegative integer, then there exists a relation between $f(x)$ and the solutions at the singularity 1. This relation comes from the behaviour of the nonholomorphic solution at 1. It is easy to see that we can write

$$f(x) = p(x) + (1-x)^d (g(x) + h(x)\log(1-x)). \tag{1.12}$$

Here $p(x)$ is a polynomial of degree $d-1$; $g(x), \mathrm{resp}\, h(x)$ are analytic at 1 and $g(1), \mathrm{resp}\, h(1) \neq 0$.

We now introduce some more notation. Let

$$\lambda_1(a), \lambda_2(a), \ldots, \lambda_m(a)$$

denote the roots of the indicial equation (local exponents) at a, which can be determined within a nonnegative integer, i.e. are of the form

$$\lambda_1(a) + n_1, \lambda_2(a) + n_2, \ldots, \lambda_m(a) + n_m$$

with $(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m$.

We shall write for short in the sequel

$$\{\lambda_1, \lambda_2, \ldots, \lambda_m\} + (\mathbb{N}). \tag{1.13}$$

Definition 2: We put

$$\delta_a = \sum_{k=1}^{m} \lambda_k(a) - m(m-1)/2. \tag{1.14}$$

The Fuchs relation can be rewritten as

$$\sum_{a \in \mathbb{P}_1} \delta_a = -m(m-1). \tag{1.15}$$

If a differential equation is without accessory parameters, i.e. $|A| = 0$, we can manipulate its unique Riemann scheme (**where we must not forget the logarithmic solutions!**) to find the equation E and the solutions of E. Moreover, the global monodromy group of E is known but in general its computation is difficult.

1.4 The Riemann–Hilbert problem and Padé-type approximation. Let $S = 0, 1, s_3, \ldots, s_k, \infty$. Put $U = \mathbb{P}_1 - S$, choose a point $p \in U$ and let

$$\mathbf{R}\colon \pi_1(U, p) \to GL(n, \mathbb{C})$$

be a homomorphism.The Riemann–Hilbert problem (Hilbert's 21^{st} problem) asks whether there is a Fuchsian differential operator

$$\frac{dy}{dx} - \left[\sum_{i=1}^{k} \frac{A_i}{x - s_i}\right] y$$

(where $s_1 = 0, s_2 = 1$) and V_p a basis of solutions of

$$\frac{dy}{dx} - \left[\sum_{i=1}^{k} \frac{A_i}{x - s_i}\right] y = 0 \tag{1.16}$$

with constant matrices A_i, such that the monodromy map

$$M\colon \pi_1(U, p) \to GL(V_p)$$

coincides with the given M for a suitable basis of V_p. For many special cases one knows that this problem has a positive answer (in particular when M is unipotent [AB]).

In fact, in this paper we only need the weaker version, which only asks for a regular singular differential equation with singular locus S and M equal to the monodromy map $\mathbf{M}$.

In this case the answer is positive (but this Fuchsian differential equation can have apparent singularities).

Let

$$F = \{f_1, f_2, \ldots, f_m\},$$

$\mathbb{C}(x)$ linearly independent analytic functions at 0 (or at infinity) of equation (1.16).

The analytic continuations of F on U generate a local system of rank m over $\mathbb{C}$, therefore these analytic continuations of F at 0 (around 1, say) give for $1 \leq k \leq m$ a new basis

$$f_k^1, f_k^2, \ldots, f_k^m.$$

(We put $f_k^1 = f_k$.)

One can construct a matrix

$$Q = (f_k^l)_{\substack{1 \leq k \leq m \\ 1 \leq l \leq m}}$$

whose columns are the solutions of equation (1.16).

Using the weaker form of the Riemann–Hilbert problem we can obtain a Fuchsian system whose solutions are given by the columns of Q.

To construct this matrix we shall use Levelt's basis.

This system is equivalent to a linear Fuchsian differential equation of order m, $L(y) = 0$ whose set of regular singular points $S' = S \cup A$ is obtained by use of the cyclic vector's lemma.

In general, the application of this lemma implies that we must add to S a set A of apparent singularities.

Now consider the linear form

$$R^1(x) = \sum_{k=1}^{m} A_k(x) f_k(x)$$

where, for $1 \leq k \leq m$, $A_k(x)$ denote polynomials of degree n_k.

The analytic continuation of $R^1(x)$ yields m linear forms $(R^1, R^2, \dots, R^m)$ which satisfy also a linear Fuchsian differential equation $L_P(y) = 0$. This differential equation depends on

$$P = \{A_1, \dots, A_m\}.$$

The set of regular singular points of $L_P(y)$ is also S (but we must add new apparent singularities). Now suppose that linear forms $(R^1, R^2, \dots, R^m)$ satisfy some conditions at the singularities (Padé-type conditions). Then we can show that L and L_P are without accessory parameters and that the operator L_P is unique.

The remainder $R^1(x)$ and the set of polynomials P are completely determined by these conditions.

We shall now apply this construction to the polylogarithmic system of functions.

2. Analytic construction of linear forms of polylogarithmic functions

2.1 Application to the polylogarithmic case. The classical polylogarithm for $k \in \mathbb{N}^*$ is defined as

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad x \in \mathbb{C}, \quad |x| \leq 1 \tag{2.1}$$

and has an analytic continuation to the cut plane X.

In the case $k = 1$, one recognizes the power series expansion of $-\log(1-x)$.

These polylogarithmic functions $Li_k(x)$ have an analytic continuation to X and may be conceived of as a 'multivalued' function on Z' (i.e. a function on W, the universal covering of Z').

Let us recall the following integral formulae,

$$Li_1(x) := -\log(1-x) = \int_0^x \frac{dt}{1-t},$$

and for the *higher logarithm*

$$Li_{k+1}(x) := \int_0^x \frac{Li_k(t)}{t} dt.$$

The relation

$$\frac{Li_k(x)}{x} =_{k+1} F_k \left({1,\ldots,1,\ldots 1 \atop 2,2,\ldots,2} | x \right) \tag{2.2}$$

between $Li_k(x)$ and the hypergeometric power series is also useful.

Now, the $m+1$ functions

$$1, \log(1-x), \ldots, Li_m(x)$$

are $\mathbb{Q}(x)$ linearly independent, therefore the local system

$$\mathcal{PL}_i(m) =: \mathbb{C}(x)\{\log(1-x), \ldots, Li_m(x)\}$$

is of rank $m+1$ over $\mathbb{C}(x)$.

The monodromy group of this local system is well known; it is, in particular, unipotent.

To find an "adapted" basis of this local system we use analytic continuation of $Li_1, Li_2, \ldots, Li_m(x)$ along loops γ_1 and γ_0 based in a vicinity of 1 resp 0.

We use the loops

$$\gamma_1, \gamma_0 \circ \gamma_1, \ldots, \gamma_0^{m-2} \circ \gamma_1, \gamma_0^{m-1} \circ \gamma_1$$

to obtain the following matrix of "periods":

$$\Lambda(x) = \begin{pmatrix} 1 & Li_1(x) & \cdots Li_m(x) \\ 0 & 2i\pi \cdots & 2i\pi \log^{(m-1)} x/(m-1)! \\ & \ddots & \\ 0 & 0 & (2i\pi)^{m-1}(2i\pi)^{m-1} \log x \\ 0 & \cdots & 0 \cdots (2i\pi)^m \end{pmatrix}. \tag{2.3}$$

The second row is a result of the monodromy transform of the first row along a loop circling 1; the third row results from the monodromy transform of the second around 0 etc....

This $(m+1) \times (m+1)$ matrix can be view as a multivalued $GL_{m+1}(\mathbb{C})$-valued matrix on $\mathbb{P}_1(\mathbb{C}) - \{0, 1, \infty\}$.

Remark 2: In fact, the local system $\mathcal{PL}_i(m)$ is $\{C_1(x), C_2(x), \ldots, C_{m+1}(x)\}$ where $C_k, 1 \leq k \leq (m+1)$ denote the columns of $\Lambda(x)$.

Let us consider $m+1$ polynomials

$$A_0(x), A_1(x), \ldots, A_m(x)$$

such that $A_m(x)$ is not zero and, for $0 \leq k \leq m$, $\deg A_k(x) = n_k$.

The linear form

$$R_0(x) = A_0(x) + \sum_{k=1}^{m} A_k(x) Li_k(x) \tag{2.4}$$

gives rise to the m new linear forms $R_1(x), R_2(x), \ldots, R_m(x)$,

$$\begin{pmatrix} R_0(x) \\ R_1(x) \\ \vdots \\ \vdots \\ R_m(x) \end{pmatrix} = \Lambda(x) \begin{pmatrix} A_0(x) \\ \vdots \\ A_k(x) \\ \vdots \\ A_m(x) \end{pmatrix}. \tag{2.5}$$

It is important to note that $R_m(x) = (2i\pi)^m A_m(x)$.

These linear forms, which depend on

$$\sigma = \sum_{k=0}^{m} (n_k + 1) - 1 \tag{2.6}$$

parameters, give rise to a new local system constructed as in (2.3), the first row of which is

$$\begin{aligned}(A_m(x), A_m(x)Li_1(x) + A_{m-1}(x), A_m(x)Li_2(x) \\ + A_{m-1}(x)Li_1(x) + A_{m-2}(x), \ldots, R_0(x)).\end{aligned}$$

This local system depends on the polynomials $A_k(x)$, $0 \leq k \leq m$.

The rank of this local system over $\mathbb{C}[x]$ is also $m+1$. It yields a family of linear differential Fuchsian operators $\mathcal{L}$ of order at least $m+1$ such that for $0 \leq k \leq m$,

$$L \in \mathcal{L} \Leftrightarrow L(R_k(x)) = 0.$$

2.2 A general Padé-type construction. Let us now consider the linear forms

$$R_0(x) = A_0(x) + \sum_{k=1}^{m} A_k(x) Li_k(x) \tag{2.7}$$

and

$$R_1(x) = 2i\pi \sum_{k=1}^{m} A_k(x) \log^{k-1}(x)/(k-1)! \tag{2.8}$$

where the polynomials $A_0(x), \ldots, A_k(x)$ are of degree n_m.

Let us suppose that $n_m \leq n_{m-1} \leq \cdots \leq n_0$.

We put

$$\sigma = \sum_{k=0}^{m}(n_k + 1) - 1.$$

We give here a solution to the following problem.

Given m positive integers $\sigma_0, \sigma_1, e_1, \ldots, e_{m-1}$, can we find polynomials $A_0(x), \ldots, A_k(x)$ such that the functions

$$R_0(x) = O(x)^{\sigma_0}, \tag{2.9}$$

$$R_1(x) = O(1-x)^{\sigma_1}, \tag{2.10}$$

and such that

$$R_k(x) = O(x)^{e_k}, \tag{2.11}$$

where

$$\sigma_0 + \sigma_1 + \sum_{k=1}^{m-1} e_k \geq \sigma? \tag{2.12}$$

The answer is given by linear algebra. We shall show that the $\mathbb{C}(x)(\frac{d}{dx})$ ideal I_L of differential operators L such that for $0 \leq k \leq m$,

$$L(R_0(x)) = 0 \cdots L(R_k(x)) = 0,$$

is "rigid", i.e. the operator L is completely determined by its local data (i.e. its Riemann scheme).

Moreover, the construction of this operator L gives almost without calculation effective formulae for the polynomials $A_k(x)$ $(1 \leq k \leq m)$ and the remainder $R_0(x)$ (depending only on the given data, namely $\sigma_0, \sigma_1, e_1, \ldots, e_{m-1}$ and the degrees $n_0, n_1, \ldots, n_m$ of the polynomials $A_k(x)$).

We can now formulate our main results.

THEOREM 2: *Under the assumptions*

$$n_m \le n_{m-1} \le \cdots \le n_0, \tag{2.13}$$

the Riemann symbol related to this equation is

$$R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & -n_0 & 0 & \\ e_1 & -n_1 & 1 & \\ e_2 & & & \\ \vdots & \vdots & \vdots & |x \\ & \vdots & & \\ e_{m-1} & & m-1 & \\ 0 & -n_m & \sigma_1 & \end{pmatrix}. \tag{2.14}$$

For $0 \le k \le m$, *the* $R_k(x)$ *are solutions of*

$$\theta(\theta-\sigma_0)\prod_{k=1}^{m-1}(\theta-e_k) - x\prod_{j=0}^{m}(\theta-n_j)(\theta+\sigma_0) = 0.$$

If $\sigma \le \sigma_0$ *and* $n_m \le n_{m-1} \le \cdots \le n_0$, *we have the following perfect Padé approximation, i.e.* $\sigma = \sigma_0$ *and within a multiplicative normalization constant*

(2.15)

$$A_m(x) = x^{n_m}{}_{m+1}F_m\left(\begin{matrix} -n_m, \sigma_0-n_m, e_1-n_m, \ldots, e_{m-1}-n_m \\ n_m-n_0+1, n_m-n_1+1, \ldots, n_m-n_{m-1}+1 \end{matrix} \,|1/x\right).$$

This polynomial belongs to $\mathbb{Z}[x]$ *and, if we set* $d_n = \mathrm{lcm}(1,2,\ldots n)$,

$$d_n^{m-k}A_{m-k}(x) \in \mathbb{Z}[x]$$

for $1 \le k \le m$.

Here, within the same multiplicative normalization constant

$$A_{m-k}(x) = \sum_{n=0}^{n} c^{(k)}(n+t)|_{t=0}x^n,$$

where $c(n)$ *denotes the coefficient of degree* n *in the polynomial* $A_m(x)$ *and, as usual,* $c^{(k)}$ *denotes the derivative of order* k *with respect to* t.

$$R_0(x) = D_n x^{\sigma_0}{}_{m+1}F_m\left(\begin{matrix} \sigma_0-n_0, \sigma_0-n_1, \ldots, \sigma_0-n_m \\ 1+\sigma_0, \ldots, 1+\sigma_0 \end{matrix} \,|x\right). \tag{2.16}$$

Here, D_n *denotes the multiplicative constant which depends on the integral formula used to write the remainder.*

For instance, use of the Euler–Riemann multiple integral for (2.16) gives

$$D_n = d_n^m \frac{\prod_{j=1}^m \Gamma(\sigma_0 - n_j)\Gamma(1+n_j)}{\Gamma(1+\sigma_0)^m}.$$

For an equivalent construction of Hermite–Padé approximations which uses "Nikishin systems" [Ni], see [Sor], [Ne3].

Remark 3: This differential equation permits us to guess the polynomials and the remainder of this Padé approximation using suitable Meijer-G functions.

The solutions of this differential equation can be written in terms of Meijer-G functions.

If we put $n_0 = n_1 = \cdots = n_m = n$, we will have

$$\begin{aligned} &\underline{a} = (1-\sigma_\infty, 1+n, \ldots, 1+n), \quad \underline{b} = (\sigma_0, 0, \ldots, 0), \\ R_0(x) &= G^{1,m+1}_{m+1,m+1}(\underline{a}, \underline{b}, (-1)^m x) \\ &= \frac{1}{2i\pi}\int_{L_1} \frac{\Gamma(\sigma_0 - s)\Gamma(s-\sigma_\infty)\Gamma(s-n)^m}{\Gamma(1-\sigma_0+s)\Gamma(1+s)^m}(-x)^s ds \end{aligned}$$

(within a multiplicative constant).

The curve L_1 may be chosen to pass from $-\infty$ to $-\infty$, encircling each of the poles of the function $\Gamma(\sigma_0 - s)$, in the positive direction, but not including any of the poles of $\Gamma(s-\sigma_\infty)$ and $\Gamma(s-n)$.

Remark 4: For $\sigma_1 \geq 1$, we can choose the following formula for the remainder:

$$\begin{aligned} R_0(x) = &\frac{\prod_{k=0}^m \Gamma(\sigma_0 - d_k)}{\prod_{k=0}^{m-1} \Gamma(1+\sigma_0 - e_k)\Gamma(1+\sigma_0)} \\ &\times \frac{1}{2i\pi}\int_L \frac{\prod_{k=0}^m \Gamma(\sigma_0 - d_k + t)\Gamma(-t)(-x)^t dt}{(\prod_{k=1}^{m-1}\Gamma(1+\sigma_0 - e_k + t))\Gamma(1+\sigma_0+t)}. \end{aligned}$$

We can also write $R_0(x)$ as the Euler–Riemann multiple integral but with another normalisation constant.

For these choices the polynomials $A_{m-k}(x)$ are exactly those given by the previous theorem (without normalisation constant!).

Proof: Let us consider the linear form

$$R_0(x) = A_0(x) + \sum_{k=1}^m A_k(x) Li_k(x).$$

We put $\mathrm{Ord}_0\, R_m(x) = \sigma_0$.

Linear algebra shows that there exist $A_k(x)$, $0 \le k \le m$ such that

$$\sigma_0 + \sigma_1 + \sum_{k=1}^{m-1} e_k \ge \sigma.$$

Using (2.5), we see that this form is related to the other linear forms obtained by use of analytic continuation of $R_0(x)$ along loops based in a vicinity of 1 resp. 0 (the monodromy around 1 and 0).

These analytic continuations permit us to introduce the local system of rank $m+1$ generated by (2.5); then

$$R_0(x), \dots, R_m(x)$$

(the so-called "Levelt basis" [Le]) are sections at 0 of this local system and also solutions of a Fuchsian differential equation L of order at least $m+1$.

In particular, we can conclude that the polynomial $A_m(x)$ satisfies $L(A_m(x)) = 0$.

Thanks to Levelt's study [Le], we can compute the exponents at the singularities 0, ∞, 1.

(These Levelt bases are invariant under the local monodromy operator.)

Consider the behaviour of the local system

$$(R_0(x), \dots, R_m(x))$$

at 0. Since

$$R_j(x), \quad 1 \le j \le m-1$$

are logarithmic lower bounds the exponents at 0 are given by

$$(\sigma_0, e_1, \dots, e_{m-1}, 0);$$

here $e_1, \dots, e_{m-1}$ denote positive integers which are lower bounds for the exponents.

According to (1.14) we obtain

$$\delta_0 = \sigma_0 - m(m-1)/2 + \sum_{k=1}^{m-1} e_k.$$

Consider the behaviour of the local system

$$(R_0(x), \dots, R_m(x))$$

at 1. We use the fact that the analytic continuation of

$$(R_0(x), R_1(x), \dots, R_{m-1}(x))$$

is holomorphic, i.e. the m initial value problem at 1 is solvable (HF_5) [Le] to obtain

$$(0, 1, \dots, m-1, \sigma_1) + (\mathbb{N})$$

and

$$\delta_1 = \sigma_1 + m + k_1.$$

But by a deeper use of the monodromy, we obtain a relation between the exponent of the logarithmic solution at 1 and σ_1.

We observe that we can write $Li_k(x)/x$ as a particular hypergeometric series, therefore an analytic continuation of $Li_k(x)$ at 1 can be written

$$Li_k(x) = Q_0^k(x) + (1-x)^{k-1}(Q_1^k(x) + Q_2^k(x)\log(1-x)) \tag{2.17}$$

where $Q_0^k(x)$ is a polynomial of degree $k-1$, $Q_1^k(x), Q_2^k(x)$ power series analytic in 1 and $Q_1^k(1)$ resp $Q_2^k(1) \neq 0$.

It follows that the analytic continuation of $R_0(x)$ at 1 can be written

$$\begin{aligned} R_0(x) =& S(x) + \sum_{k=1}^{m}(1-x)^{k-1}Q_1^k(x)A_k(x) \\ &+ \left(\sum_{k=1}^{m}(1-x)^{k-1}Q_2^k(x)A_k(x)\right)\log(1-x). \end{aligned} \tag{2.18}$$

Here, $S(x)$ denotes a polynomial of degree at most $n+m-1$.

Monodromy around the singularity 1 gives

$$R_0(x) \to R_0(x) + 2i\pi\left(\sum_{k=1}^{m}(1-x)^{k-1}Q_2^k(x)A_k(x)\right).$$

But since the monodromy around 1 yields

$$R_0(x) \to R_0(x) + R_1(x),$$

if we put $r_1(x) = R_1(x)/2i\pi$ we obtain

$$r_1(x) = \sum_{k=1}^{m}(1-x)^{k-1}Q_2^k(x)A_k(x). \tag{2.19}$$

Consequently, the exponent related to $R_1(x)$ is $\sigma_1 + (\mathbb{N})$.

Moreover, since $Q_2^k(1) \neq 0$ then $A_1(1) = 0$ if and only if $\sigma_1 \geq 1$.

Consider the behaviour of the local system

$$(R_0(x), \dots, R_m(x))$$

at infinity.

At "∞", to study the analytic continuation of the functions $R_j(x), 1 \leq j \leq m-1$ we have to suppose that $n_m \leq n_{m-1} \leq \cdots \leq n_0$ (since to obtain exponents at this point it is easy to see that we must have $-n_0 \leq -n_1 \leq \cdots \leq -n_m$).

Therefore, lower bounds $+(\mathbb{N})$ are equal to $-n_1, -n_2, \dots, -n_m$ and there exists only a problem for the analytic continuation of $R_0(x)$.

To find the analytic continuation at ∞ of $R_0(x)$, we must use (connexion formulae) [Oe] for $Li_k(x)$, namely

$$Li_k(x) + (-1)^k Li_k(1/x) = -\frac{(2i\pi)^k}{k!} B_k(\log' x/2i\pi) \tag{2.20}$$

where $B_k(x)$ denotes the Bernoulli polynomial of degree k and $(\log' x)$ the branch of log whose imaginary part is in $]0, 2\pi[$.

If we put

$$(1/x)^{\sigma_\infty} V(1/x) = -A_0(x) + \sum_{k=1}^{m} A_k(x)(-1)^{k+1} Li_k(1/x), \tag{2.21}$$

where $V(1/x)$ is analytic and not zero at ∞, and

$$U(1/x) = -\sum_{k=1}^{m} A_k(x) \frac{(2i\pi)^k}{k!} B_k(\log' x/2i\pi), \tag{2.22}$$

then if we put

$$W(1/x) = (1/x)^{\sigma_\infty} V(1/x) + U(1/x) \tag{2.23}$$

we easily get exponents at ∞, namely

$$(\sigma_\infty, -n_1, -n_2, \dots, -n_m) + (\mathbb{N})$$

with

$$-n_0 \leq \sigma_\infty \leq 0, \tag{2.24}$$

$$\delta_\infty = \sigma_\infty - \sum_{k=1}^{m} n_k - m(m-1)/2 + k_\infty.$$

The Fuchs relation gives

$$-m(m+1)/2 = \Big(\sigma_0 - m(m+1)/2 + \sum_{k=1}^{m-1} e_k + (\sigma_\infty + n_0) - \sum_{k=0}^{m} n_k - m(m+1)/2 + k_\infty\Big) + (\sigma_1 - m + k_1) + S_a$$

or

$$(\sigma_\infty + n_0) + \sigma_0 + \sigma_1 + k_0 + k_1 + k_\infty + S_a = \sum_{k=0}^{m} n_k + m,$$

which yields with the conditions $\sigma_0 + \sigma_1 + \sum_{k=1}^{m-1} e_k \leq \sigma$

$$(\sigma_\infty + n_0) + \sum_{k=1}^{m-1} e_k + k_1 + k_\infty + S_a \leq 0.$$

Since $\sigma_\infty \geq -n_0$ then

$$k_1 = k_\infty = S_a = 0,$$

which establishes that the operator L is hypergeometric of order $m+1$.

The Riemann scheme of L is in this general case

$$R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & \sigma_\infty & 0 & \\ e_1 & -n_0 & 1 & \\ e_2 & -n_1 & 2 & |x \\ \vdots & \vdots & \vdots & \\ e_{m-1} & \vdots & m-1 & \\ 0 & -n_m & \sigma_1 & \end{pmatrix}. \tag{2.25}$$

This Riemann scheme can also be identified with

$$x^{\sigma_0} R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & \sigma_\infty - \sigma_0 & 0 & \\ e_1 - \sigma_0 & \sigma_0 - n_0 & 1 & \\ e_2 - \sigma_0 & \sigma_0 - n_1 & 2 & |x \\ \vdots & \vdots & \vdots & \\ e_{m-1} - \sigma_0 & \vdots & m-1 & \\ -\sigma_0 & \sigma_0 - n_m & \sigma_1 & \end{pmatrix}$$

which is equivalent to (2.25).

Since the remainder $R_0(x)$ belongs to the exponent "σ_0" of the previous Fuchsian local system, $R_0(x)$ can be written

(2.26)

$$R_0(x) = \frac{\prod_{j=1}^m \Gamma(\sigma_0 - n_j)\Gamma(1+n_j)}{\Gamma(1+\sigma_0)^m} x^{\sigma_0} \times {}_{m+1}F_m \left(\begin{matrix} \sigma_0 - n_0, \sigma_0 - n_1, \sigma_0 - n_2, \ldots, \sigma_0 - n_m \\ 1+\sigma_0, 1+\sigma_0 - e_1, 1+\sigma_0 - e_2, \ldots, 1+\sigma_0 - e_{m-1} \end{matrix} \Big| x \right),$$

also written

$$\int_{[0,1]^m} \prod_{k=0}^m \frac{t_k^{\sigma_0 - n_k - 1}(1-t_k)^{n_k+e_k-1}}{(1-t_1 t_2 \cdots t_m)^{\sigma_0 - n_m}} dt_1 \cdots dt_m. \tag{2.27}$$

The use of the last row of (2.18) (or monodromy around "1"), shows that $A_m(x)$ is a solution of the same hypergeometric differential equation.

To write this polynomial we use the Fuchsian local system for the local variable $1/x$ at infinity. The Riemann scheme can be identified with

$$\left(\frac{1}{x}\right)^{-n_m} R \left(\begin{matrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_\infty + n_m & \sigma_0 - n_m & 0 & \\ n_m - n_0 & e_1 - n_m & 1 & \\ & e_2 - n_m & 2 & |1/x \\ \vdots & \vdots & \vdots & \\ n_m - n_{m-1} & e_{m-1} - n_m & m-1 & \\ 0 & -n_m & \sigma_1 & \end{matrix} \right).$$

It follows easily that

(2.28)

$$A_m(x) = x^{n_m} {}_{m+1}F_m \left(\begin{matrix} -n_m, \sigma_0 - n_m, e_1 - n_m, \ldots, e_{m-1} - n_m \\ 1 + n_0 - n_m, 1 + n_1 - n_m, \ldots, 1 + n_{m-1} - n_m \end{matrix} \Big| 1/x \right).$$

Now if $\sigma_0 \geq \sigma$, then $\sigma_1 = e_1 = \cdots = e_{m-1} = 0$ and we obtain the formulae for the Padé approximation of the first kind.

If $n_0 = n_1 = \cdots = n_m = n$, we have

$$A_m(x) = x^n {}_{m+1}F_m \left(\begin{matrix} \sigma_0 - n, -n, -n, \ldots, -n \\ 1, 1, 1, \ldots, 1 \end{matrix} \Big| 1/x \right) \tag{2.29}$$

or, for arithmetic applications,

$$A_m(x) = \sum_{k=0}^n (-1)^{k(m+1)} \binom{n}{k}^m \binom{\sigma_0 - 1}{k} x^{n-k} \in \mathbb{Z}[x]. \tag{2.30}$$

Moreover, for $1 \leq l < m$ the polynomials $A_{m-l}(x)$ (within the same multiplicative constant) are given by the Frobenius method of derivation with respect to the parameters [In].

3. Well-poised hypergeometric polynomials and Rivoal's construction

We establish in this section a relation between our constructions of Fuchsian linear differential equations and the nice results of Ball, Rivoal [Ba], [Ri] and Zudilin, [Zu1], [Zu2] on the arithmetic nature of the odd zeta values.

In order to get arithmetic results, these authors require that for $0 \leq k \leq m$ the polynomials $A_k(x)$ are subject to the "almost reciprocal condition", i.e.

$$A_k(x) = (-1)^{\sigma-k} x^n A_k(1/x), \tag{3.1}$$

where, as in the previous section, $\sigma = (m+1)(n+1) - 1$ and $R_0(x) = O(x)^{\sigma_0}$.

The following theorem provides a natural construction via linear differential equations of their linear forms involving polylogarithms and odd zeta values.

Let us recall the following definitions.

Definition 3: If the parameters in the hypergeometric power series

$$ {}_{m+1}F_m \left(\begin{matrix} a_0, a_1, \dots a_m \\ b_1, \dots, b_m \end{matrix} | x \right) $$

satisfy the relations

$$a_0 + 1 = a_1 + b_1 = \cdots = a_m + b_m, \tag{3.2}$$

the series is said to be well-poised.

The series is said to be very well-poised if it is well-poised and, moreover, we have the relation

$$a_1 = \frac{1}{2} a_0 + 1.$$

The series is said to be nearly poised if all but one of the pairs of parameters have the same sum.

The very well-poised condition permits us to have a shifting in the relation (3.16). (See [Fi1].)

3.1 Well-poised polynomials and hypergeometric differential equations.

Definition 4: Let $\mathcal{H} = \mathcal{H}(A, B)$ denote a linear Fuchsian hypergeometric differential of order m. The operator $\mathcal{H}$ is said to be "well-poised" if any solution $Z(x)$ of $\mathcal{H} = 0$ satisfies $\mathcal{H}(x^n Z(1/x)) = 0$.

THEOREM 3: *Let $\mathcal{H}$ denote a well-poised hypergeometric operator. Then there exists a non zero polynomial $P_n(x)$ of degree n satisfying:*

(i) $\mathcal{H}((P_n)(x)) = 0$.

$$x^n P_n(1/x) = k P_n(x) \quad (k \in \mathbb{C}^*).$$

(ii) *If $m+1$ denotes the order of $\mathcal{H}$ and*

$$expS_0 = \{1-b_0, 1-b_1, 1-b_2, \dots, 1-b_m\}, \quad expS_\infty = \{a_0, a_1, a_2, \dots, a_m\}$$

denotes the set of exponents at the singular points 0 resp ∞ of $\mathcal{H}$, then for $0 \le k \le m$, $b_0 = 1$, the "well-poised" relations

$$1 - n = a_k + b_k$$

are satisfied.

(iii) *Suppose that there exists another well-poised holomorphic solution $R_0(x)$ of $\mathcal{H}$ of the form $R_0(x) = x^{\sigma_0} f(x)$, $f(0) \neq 0$ where $\sigma_0 = 1 - b_1 \ge n + 1$. Then*

$$R_0(x) = x^{\sigma_0} {}_{m+1}F_m \left(\begin{matrix} 2\sigma_0 - n, \sigma_0 - n, \sigma_0 - b_2, \dots, \sigma_0 - b_m \\ 1 + \sigma_0, \sigma_0 + 1 + b_2, \dots, \sigma_0 + 1 + b_m \end{matrix} | x \right)$$

(within a multiplicative constant). Using the operator $\theta = x\frac{d}{dx}$ we can write this expression

$$x^{\sigma_0} \frac{(\theta + \sigma_0 + 1)_{(\sigma_0 - n)}}{(\sigma_0 + 1)_{(\sigma_0 - n)}} \left[{}_mF_{m-1} \left(\begin{matrix} \sigma_0 - n, \sigma_0 - b_2, \dots, \sigma_0 - b_m \\ \sigma_0 + 1 + b_2, \dots \sigma_0 + 1 + b_m \end{matrix} | x \right) \right],$$

where $(a)_n$ denotes the Pochammer symbol.

The rank of the local system related to $\mathcal{H}$ over $\mathbb{C}[x]$ is m.

Proof: Let us consider the hypergeometric operator $\mathcal{H}$ whose solutions at 0 are given by the local system S_0 of order $m + 1$.

Suppose that a Riemann scheme related to this local system is given by

$$R_1\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & a_0 & 0 & \\ 1-b_1 & a_1 & 1 & \\ \vdots & \vdots & 2 & |x \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ 1-b_m & a_m & d & \end{pmatrix}.$$

Since by assumption

$$\mathcal{H}(Z)(x)=0 \Leftrightarrow \mathcal{H}(x^nZ(1/x))=0,$$

the local system related to the solutions $x^nZ(1/x)$ gives rise to a local system at infinity. Then

$$ExpS_\infty=\{-n,1-n+b_1,\dots,1-n+b_m\}=\{a_0,a_1,\dots,a_m\}.$$

Comparing these two sets of exponents permits us to obtain the well-poised relations $n+a_0=0$ and, for $1\le k\le m$, $1-b_k=n+a_k$, i.e. $a_0=-n$, $1\le k\le m$, $a_k+b_k=1-n$. ∎

At infinity one exponent is a negative integer $-n$.

The Frobenius method ([In] pages 397–398) shows that the analytic function at infinity satisfying $\mathcal{H}(Z(x))=0$ can be written

$$Z(x)=\sum_{k=0}^{n}c_k(1/x)^{-k}=\sum_{k=0}^{n}c_kx^k$$

with $c_n\neq 0$, and this proves that $Z(x)$ is a polynomial.

Furthermore, if we assume that $1-b_1=\sigma_0$ is a positive integer with $\sigma_0\ge n+1$, then this local system can be rewritten

$$x^{\sigma_0}R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & 2\sigma_0-n & 0 & \\ 0 & \sigma_0-n & 1 & \\ 1-b_2-\sigma_0 & \sigma_0-n-b_2 & 2 & |x \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ 1-b_m-\sigma_0 & \sigma_0-n-b_m & d & \end{pmatrix}.$$

This yields the expression of $R_0(x)$ in a hypergeometric power series.

Using the following 'contiguity relation' for a hypergeometric power series (1.2)

$$(\theta + a_1)_{m+1}F_m\left(\begin{matrix} a_1, a_2, \dots, a_{m+1} \\ b_1, b_2, \dots, b_m \end{matrix}\middle| x\right) = a_1 \cdot {}_{m+1}F_m\left(\begin{matrix} a_1+1, a_2, \dots, a_{m+1} \\ b_1, b_2, \dots, b_m \end{matrix} |x\right)$$

$\sigma_0 - n$ times, we obtain

$$R_0(x) = x^{\sigma_0}\frac{(\theta+\sigma_0+1)_{(\sigma_0-n)}}{(\sigma_0+1)_{(\sigma_0-n)}}\left[{}_mF_{m-1}\left(\begin{matrix} \sigma_0 - n, \dots, \sigma_0 - b_2, \sigma_0 - b_m \\ , \sigma_0+1+b_2, \sigma_0+1+b_m \end{matrix} |x\right)\right].$$

Since the previous hypergeometric power series satisfies a Fuchsian differential equation of order m, the rank of the local system related to $\mathcal{H}$ over $\mathbb{C}[x]$ is m.

Remark 5: This relation shows that if $S(a_1, a_2, \dots, a_{m+1}, b_1, \dots, b_m)$ denotes the linear space of solutions of the general hypergeometric differential equation, the first order differential operator $(\theta + a_1)$ sends

$$S(a_1, a_2, \dots, a_{m+1}, b_1, \dots, b_m)$$

into

$$S(a_1+1, a_2, \dots, a_{m+1}, b_1, \dots, b_m).$$

Since the hypergeometric operator $\mathcal{H}(a_1, a_2, \dots, a_{m+1}, b_1, \dots, b_m)$ satisfies

$$\begin{aligned}(\theta + a_1)&\mathcal{H}(a_1, a_2, \dots, a_{m+1}, b_1, \dots, b_m)\\ &= \mathcal{H}(a_1+1, a_2, \dots, a_{m+1}, b_1, \dots, b_m)(\theta + a_1),\end{aligned}$$

the operator $(\theta + a_1)$ is a step-up operator for a_1.

The following theorem gives an application of such a situation.

THEOREM 4: *Suppose that $R_0(x)$ satisfies a well-poised hypergeometric differential equation of order $m+2$. Then the local system related to $R_0(x)$ over $\mathbb{C}[x]$ is of rank m.*

The Riemann symbol is

$$R\left(\begin{matrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & \sigma_0 - n_m & 0 & \\ \sigma_0 & -n_0 & 1 & \\ n_m - n_0 & & & \\ \vdots & \vdots & \vdots & |x \\ & \vdots & & \\ n_m - n_{m-2} & & m-1 & \\ n_m - n_{m-1} & -n_m & \sigma_1 & \end{matrix}\right). \tag{3.3}$$

$$R_0(x) = \sum_{k=1}^{m} A_k(x) Li_k(x) + A_0(x)$$

is a holomorphic solution of

$$\prod_{k=0}^{m}(\theta+n_m-n_k)(\theta-\sigma_0)-x\prod_{k=0}^{m}(\theta-n_k)(\theta+\sigma_0-n_m)=0. \tag{3.4}$$

$$R_0(x)= \tag{3.5}$$
$$C(\sigma_0)x^{\sigma_0}{}_{m+2}F_{m+1}\left(\begin{matrix}2\sigma_0-n_m,\sigma_0-n_0,\ldots,\sigma_0-n_m\\ 1+\sigma_0+n_0-n_m,\ldots,1+\sigma_0+n_{m-1}-n_m\end{matrix}\,|x\right).$$

Here, $C(\sigma_0)$ denotes an arithmetic constant of normalisation. If $\sigma_0>1+n_0$ ([Ba,Ri], [Ri,Zu]),

$$A_m(x)={}_{m+2}F_{m+1}\left(\begin{matrix}\sigma_0-n_m,-n_0,-n_1,\ldots,-n_{m-1}\\ 1-\sigma_0,1+n_0-n_m,\ldots,1+n_{m-1}-n_m\end{matrix}\,|x\right), \tag{3.6}$$

$$\sigma_1=-2\sigma_0+(m+1)+2\sum_{k=0}^{m-1}n_k-md_m\geq 1 \tag{3.7}$$

(well-poised polynomials), and if we choose σ_0 satisfying $\sigma_1\geq 1$, then $A_1(1)=0$. In particular, $R_0(1)$ gives $\mathbb{Q}$ linear forms in even or odd zeta values depending on the evenness or oddness of m, $m\geq 2$.

If we put $d_n=\operatorname{lcm}(1,2,\ldots,n)$ we obtain

$$d_n^m R_m(1)\in\mathbb{Z}\zeta(m)+\mathbb{Z}\zeta(m-1)+\cdots+\mathbb{Z}\zeta(3)+\mathbb{Z}$$

for m odd [Ba,Ri].

Remark 6: If $n_0=n_1=\cdots=n_m=n$, the hypergeometric operator L of order $m+3$ related to the nearly-poised polynomial is

$$\theta^{m+1}(\theta-\sigma_0)\Big(\theta-\frac{n}{2}-1\Big)-x(\theta-n)^{m+1}(\theta+\sigma_0-n)\Big(\theta+\frac{n}{2}+1\Big)=0.$$

The nearly-poised polynomial related to this approximation is given by [Fil]

$$A_m(x)={}_{m+3}F_{m+2}\left(\begin{matrix}\sigma_0-n,-\frac{n}{2}+1,-n,-n,\ldots,-n\\ 1-\sigma_0,-\frac{n}{2},1,\ldots,1\end{matrix}\,|x\right)$$

and

$$-\frac{n}{2}A_m(x)=\Big(\theta-\frac{n}{2}\Big)\left[{}_{m+2}F_{m+1}\left(\begin{matrix}\sigma_0-n,-n,-n,\ldots,-n\\ 1-\sigma_0,1,\ldots,1\end{matrix}\,|x\right)\right].$$

Proof of Theorem 4: We use the result of the theorem (2.2) where one considers the remainder (2.16).

Let us suppose the Fuchsian differential equation of the theorem (2.2) well-poised and of order $m+2$. The previous theorem shows that the local system related to this equation is of rank m, hence the polylogarithm function of maximal weight related to this local system is $Li_m(x)$.

The proof of the theorem (2.2) shows (using monodromy at 1) that the polynomial $A_m(x)$ satisfies the same Fuchsian differential equation as $R_0(x)$ and this polynomial is well-poised.

Now by application of Theorem 3, we can conclude that the Riemann scheme related to this hypergeometric equation is given by

$$R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & \sigma_\infty & 0 & \\ e_0 & -n_0 & 1 & \\ e_1 & -n_1 & 2 & |x \\ \vdots & \vdots & \vdots & \\ e_{m-1} & \vdots & m-1 & \\ 0 & -n_m & \sigma_1 & \end{pmatrix}$$

and we deduce that the hypergeometric well-poised polynomial is

$$A_m(x) = \\ x^{d_m}\, {}_{m+2}F_{m+1}\left(\begin{matrix}\sigma_0-n_m, \sigma_0-n_0, \sigma_0-n_1, \ldots \sigma_0-n_{m-1} \\ 1-\sigma_0, 1+n_0-n_m-\sigma_0, \ldots, 1+n_{m-1}-n_m-\sigma_0\end{matrix}\,\Big|1/x\right).$$

This polynomial can also be written as

$$C(m,n)\,{}_{m+2}F_{m+1}\left(\begin{matrix}\sigma_0-n_m, -n_0, -n_1, \ldots -n_{m-1} \\ 1-\sigma_0, 1+n_0-n_m, \ldots, 1+n_{m-1}-n_m\end{matrix}\,\Big|x\right),$$

where $C(m,n)$ is a constant.

Now, comparison of terms of degree n in these two hypergeometric polynomials gives $C(m,n) = (-1)^{n_m(m+1)}$.

This yields the "well-poised" condition for the hypergeometric polynomial $A_m(x)$.

If $c_k^m(n)$ denotes the coefficient of degree k of $A_m(x)$, then this well-poised relation is equivalent to

$$c_k^m(n) = (-1)^{n_m(m+1)} c_{n-k}^m(n). \tag{3.8}$$

The other polynomials $A_{m-k}(x)$ are given using the Frobenius method by derivation with respect to the parameters.

It remains to show that if c_k^{m-k} denotes the term of degree k of $A_{m-k}(x)$, then

$$c_k^{m-k} = (-1)^{n_m(m+1)-k} c_{n-k}^{m-k}.$$

The Frobenius method of derivation gives

$$c_k^{m-k} = \frac{d^k}{dt^k}(c_{t+k}^{m-k})|_{t=0} \quad \text{and} \quad c_{n-k}^{m-k} = \frac{d^k}{dt^k}(c_{n-(t+k)}^{m-k})|_{t=0}$$

and the well-poised relation for $A_{m-k}(x)$ follows easily.

Finding the remainder formula for $R_0(x)$ is easy by use of the Riemann scheme.

We find (within a multiplicative constant)

$$R_0(x) = x^{\sigma_0}{}_{m+2}F_{m+1}\left(\begin{matrix} 2\sigma_0 - n_m, \sigma_0 - n_0 \dots, \sigma_0 - n_m \\ 1+\sigma_0+n_0-n_m, \dots, 1+\sigma_0+n_{m-1}-n_m \end{matrix} \Big| x\right).$$

Now the Fuchs relation must be satisfied, i.e.

$$2\sigma_0 - 2\sum_{k=0}^{m} n_k + mn_m + \sigma_1 = m+1.$$

The condition $A_1(1) = 0$ is satisfied if and only if $\sigma_1 \geq 1$.

We can choose σ_1 such that (3.7) is satisfied. The Frobenius method, monodromy or (in this particular polylogarithmic case), more easier, the expansion of the remainder in a sum of partial fractions [Ri, Zu], gives the other polynomials. ∎

At 1, we can write the remainder with Euler's Gamma functions

(3.9)

$$R_0(1) = \frac{\prod_{k=0}^{m}\Gamma(1+\sigma_0+n_k-n_m)}{\prod_{k=0}^{m}\Gamma(\sigma_0-n_k)\Gamma(2\sigma_0-n_m)} \int_L \frac{\prod_{k=0}^{m}\Gamma(\sigma_0-n_k+t)\Gamma(2\sigma_0-n_m+t)}{\prod_{k=0}^{m}\Gamma(1+\sigma_0+n_k-n_m+t)\Gamma(1+t)}dt.$$

The very well-poised polynomial gives a hypergeometric operator $P(D)$ of order $m+3$ such that $P(D)(R_0(x)) = 0$.

Remark 7: In [Ba] and [Ri], one puts $\sigma_0 = (r+1)n+1$.

Since $d \geq 1$, we can apply Remark 5 to find the multiplicative normalisation constant for the remainder,

$$C(\sigma) = \frac{((\sigma_0-n-1)!)^m n!^m}{(\sigma_0!)^n}.$$

In this case, the polynomials $A_{m-k}(x)$ are those given by the theorem and, for arithmetic reasons, we must find $D_n \in \mathbb{N}$ such that

$$D_n A_m(x) \in \mathbb{Z}[x].$$

In conclusion, we can take as multiplicative normalization constant

$$C(\sigma_0) = d_n^m n!^{m-1-2r} \frac{(\sigma_0 - n - 1)!(2\sigma_0 - n - 1)!}{((\sigma_0 + 1)!)^{m-1}},$$

the polynomials $A_k(x)$ being replaced by

$$d_n^{m-k} n \frac{(\sigma_0 - n - 1)!(\sigma_0 - 1)!}{n!^{2r+1}} A_k(x) \in \mathbb{Z}[x].$$

3.2 Differential equations dilogarithm and $\zeta(2)$. This is a particular case of the previous sections (see also [Beu], [Hu3], [Rh, V]).

Let us begin with simultaneous approximations of dilogarithms and logarithms and consider the two linear forms

$$R_0(x) = A_2(x)Li_2(x) + A_1(x)\log(1-x) + A_0(x), \tag{3.10}$$

$$R_1(x) = A_2(x)\log x + A_1(x). \tag{3.11}$$

We put

$$\sigma = d_0 + d_1 + d_2 + 2, \quad \mathrm{ord}_0\, R_0(x) = \sigma_0, \quad \mathrm{ord}_1\, R_1(x) = \sigma_1.$$

Using the previous study, there exist polynomials $A_0(x), A_1(x), A_2(x)$ whose degrees are given by

$$\deg A_2(x) = d_2, \quad \deg A_1(x) = d_1, \quad \deg A_0(x) = d_0$$

with $d_2 \leq d_1 \leq d_0$ such that

$$\sigma_0 + \sigma_1 + e_1 \geq \sigma.$$

The local system related to $R_0(x)$ is also hypergeometric and one can find without computation the differential equation related to this system.

Since, for instance,

$$Li_2(x) = -Li_2(1-x) + \log x \log(1-x) + \pi^2/6, \tag{3.12}$$

one can deduce that

$$R_0(x) = \Phi(1-x) + R_1(1-x)\log(1-x)$$

where $\Phi(1-x)$ is an analytic function in 1 (a particular case studied in the proof of Theorem 2).

Then the system of exponents at 1 is

$$(0, 1, \sigma_1) + (\mathbb{N}),$$
$$\delta_1 = 1 + \sigma_1 + k_1 - 3.$$

The Fuchs relation gives

$$3 = \sigma_0 - d_0 - d_1 - d_2 + \sigma_1 + 1 + S_a$$

or

$$\sigma_0 + \sigma_1 \geq \sigma = \sigma_0 + \sigma_1 + S_a + k_0 + k_\infty + k_1,$$

which forces $S_a = k_0 = k_\infty = k_1 = 0$.

The Riemann scheme of L is in this case

$$R\left(\begin{matrix} \underline{0} & \underline{\infty} & \underline{1} \\ 0 & -d_0 & 0 \\ e_1 & -d_1 & 1 \\ \sigma_0 & -d_2 & \sigma_1 \end{matrix} |x\right), \tag{3.13}$$

which gives (within a multiplicative constant) the following polynomial:

$$x^{d_2}{}_3F_2\left(\begin{matrix} -d_2, e_1 - d_2, \sigma_0 - d_2 \\ -d_2 + d_0 + 1, -d_2 + d_0 + 1 \end{matrix} |1/x\right).$$

The remainder is also equal (within a multiplicative constant) to

$$x^{\sigma_0}{}_3F_2\left(\begin{matrix} \sigma_0 - d_0, \sigma_0 - d_1, \sigma_0 - d_2 \\ 1 + \sigma_0, 1 + \sigma_0 - e_1 \end{matrix} |x\right).$$

For $d_0 = d_1 = d_2 = n$, we obtain

$$e_1 = 0, \quad \sigma_0 = 2n + 1, \quad \sigma_1 = n + 1,$$

and one recovers in this case the construction used in Apéry's construction [Ap] of rational approximations of $\zeta(2)$, namely

$$A_2(x) = x^n{}_3F_2\left(\begin{matrix} -n, -n, n+1 \\ 1, 1 \end{matrix} |1/x\right),$$

which can be written

$$A_2(x) = \sum_{l=0}^{l=n} \binom{n}{l}^2 \binom{n+l}{l} x^{n-l}.$$

The corresponding integral for the remainder is exactly that given by Beukers [Beu].

Let us recall that $a_n = A_2(1)$ and $b_n = A_0(1)$ satisfy the recurrence equation [Ap]

$$(n+1)^2 u_{n+1} - (11n^2 + 11n + 3) - n^2 u_{n-1} = 0.$$

Since $A_2(x)$ is a hypergeometric polynomial, we can find such a relation by use of Zeilberger's algorithm of creative telescoping [Ze].

4. Differential equation and approximations of $\zeta(3)$

4.1 Differential equations and simultaneous approximations of $\zeta(2)$ and $\zeta(3)$. Let us recall Beukers's method [Beu] concerning simultaneous approximations of $\zeta(2)$ and $\zeta(3)$.

Linear algebra shows that there exist four polynomials

$$A_3(x), A_2(x), A_1(x), A_0(x)$$

of degree n such that

$$R_1(x) = A_3(x)Li_2(x) + A_2(x)Li_1(x) + A_1(x),$$
$$R_2(x) = 2A_3(x)Li_3(x) + A_2(x)Li_2(x) + A_0(x),$$

satisfying $\mathrm{Ord}_0\, R_1(x) \geq 2n+1$, $\mathrm{Ord}_0\, R_2(x) \geq 2n+1$ and $A_2(1) = 0$.

Remark 8: The main idea to motivate the introduction of $R_2(x)$ comes from the Frobenius method of perturbing the power series.

With this aim we introduce the function

$$Li_k(x,s) = \sum_{n=1}^{\infty} \frac{x^{n+s}}{(n+s)^k},$$

where s denotes a 'formal' variable. Since

$$\frac{\partial Li_k(x,s)}{\partial s}|_{s=0} = Li_k(x)\log x - kLi_{k+1}(x),$$

using the function

$$R_1(x,s) = A_3(x)Li_2(x,s) + A_2(x)Li_1(x,s) + A_1(x,s)x^s$$

it is easily seen that we have

$$\frac{\partial R_1(x,s)}{\partial s}|_{s=0} = R_1(x)\log x - R_2(x)$$

with $A_1(x) = A_1(x,s)|_{s=0}$ and $A_0(x) = \frac{\partial A_1(x,s)}{\partial s}|_{s=0}$.

We now put

$$\tilde{R}_2(x) = \log x.R_1(x) - R_2(x). \tag{4.1}$$

Considering the local system $\mathbb{C}[x](Li_1(x), Li_2(x), Li_3(x))$ one sees that we can construct a linear differential operator L of order at least 4 such that, at 0,

$$\tilde{R}_2(x) = \log x.R_1(x) - R_2(x)$$

is a solution of $L = 0$.

Monodromy around 0 shows that $L(R_1(x)) = 0$.

Now if we put

$$R_3(x) = A_3(x)\log x + A_2(x),$$

then monodromy around 1 shows that $L(R_3(x)) = 0$.

Monodromy around 0 for $R_4(x) = A_3(x)$ yields $L(R_4(x)) = 0$.

We can conclude that $\tilde{R}_2(x), R_1(x), R_3(x), R_4(x)$ are linearly independent solutions at 0 of $L = 0$.

We can take the following 'Levelt' basis of solutions of L at 0,

$$R_1(x), \tilde{R}_2(x), R_3(x), R_4(x).$$

Since $\mathrm{Ord}_0\, R_1(x) \geq 2n+1$ and $\mathrm{Ord}_0\, R_2(x) \geq 2n+1$, it is easy to see that the exponents of L at 0 are $(2n+1, 2n+1, 0, 0) + (\mathbb{N})$, and that

$$\delta_0 = 4n + 2 - 6 + k_0 = 4n - 4 + k_0.$$

At "∞", one finds $(-n, -n, -n, -n) + (\mathbb{N})$, and

$$\delta_\infty = -4n - 6 + k_\infty.$$

At 1, we must verify that the three-dimensional initial-value problem is satisfied. (See Theorem 1.)

At this singularity, the operator L has two holomorphic solutions $R_3(x)$ and $R_4(x)$.

We prove that $R_1(x)$ is logarithmic and belongs to the exponent 1. Indeed

$$Li_2(x) = Q_0(x) + (1-x)(Q_1(x) + Q_2(x)\log(1-x))$$

and it suffices to remark that $A(1) = 0$.

Now $\tilde{R}_2(x)$ can be written in the form

$$\tilde{R}_2(x) = U_1(x) + (1-x)^2(U_2(x)\log(1-x) + V_2(x)),$$

where $U_1(x)$ denotes a polynomial of degree 1 and $U_2(x)$ (resp $V_2(x)$) are holomorphic at 1 with $V_2(1) \neq 0$.

It remains to verify that 2 is an exponent which comes from a holomorphic solution.

If we suppose the solution related to L at this point is logarithmic, a factor on the form $\log(1-x)^2$ should appear in the expression given by $\tilde{R}_2(x)$ ([In] page 401). It is not the case, and 2 is an exponent which comes from a holomorphic solution at 1.

We can also verify this point using monodromy. Indeed monodromy around 1 shows that $(1-x)^2 U_2(x)$ is a holomorphic solution of $L=0$ belonging at the exponent 1.

This gives using Levelt's theory the following exponents at 1 and the "3"-dimensional problem is solvable.

The exponents are

$$(0,1,2,1)+(\mathbb{N})$$

and $\delta_1 = 4-6+k_1 = -2+k_1$.

According to the Fuchs relation, this differential equation of order 4 yields

$$-12 = (4n-4+k_0)+(-4n-6+k_\infty)+(-2+k_1)+S_a$$

or $k_0+k_\infty+k_1+S_a=0$. One concludes that the Riemann scheme related to this equation is given by

$$(4.2) \qquad R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & -n & 0 & \\ 2n+1 & -n & 1 & |x \\ 2n+1 & -n & 2 & \\ 0 & -n & 1 & \end{pmatrix}.$$

It is related to the differential hypergeometric equation

$$\theta^2(\theta-2n-1)^2 - x(\theta-n)^4 = 0.$$

It is also possible to find the form of the remainder, using the Riemann scheme (see [Hu3]). One finds that $R_1(x)$ is equal (within a multiplicative constant) to

$$(4.3) \qquad x^{2n+1}{}_4F_3\left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 2n+1, 2n+1, 1 \end{matrix}\Big| x\right).$$

The normalization gives

$$\frac{(n!)^4}{((2n)!)^2} x^{2n+1}{}_4F_3\left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 2n+1, 2n+1, 1 \end{matrix}\Big| x\right).$$

A straightforward computation shows that if one puts

$$ {}_4F_3\left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 2n+1, 2n+1, 1 \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} c(n)x^n, $$

then

$$ R_2(x) = \frac{(n!)^4}{(2n!)^2} x^{2n+1} \frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} c(n+t)x^{n+t}\right]|_{t=0}, \tag{4.4} $$

which gives

$$ \begin{aligned} \tilde{R}_2(x) = x^{2n+1}\frac{(n!)^4}{(2n!)^2}\Bigg[& {}_4F_3\left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 2n+1, 2n+1, 1 \end{matrix} \middle| x\right)\log x \\ & + \sum_{k=1}^{\infty} \frac{((n+k)!)^4}{((2n+k)!)^2.k!^2}\left(\sum_{s=1}^{k} \frac{4}{n+1+s} - \frac{2}{2n+1+s} - \frac{2}{s}\right)x^k\Bigg]. \end{aligned} $$

Remark 9:

$$ {}_4F_3\left(\begin{matrix} n+1, n+1, n+1, n+1 \\ 2n+1, 2n+1, 1 \end{matrix} \middle| x\right) $$

can be written

$$ \begin{aligned} \frac{1}{n!}\prod_{k=1}^{n+1}(\theta+k)&\left[{}_4F_3\left(\begin{matrix} n+1, n+1, n+1, 1 \\ 2n+1, 2n+1, 1 \end{matrix} \middle| x\right)\right] \\ &= \frac{1}{n!}\frac{d^n}{dx^n}\left[x^n {}_3F_2\left(\begin{matrix} n+1, n+1, n+1 \\ 2n+1, 2n+1 \end{matrix} \middle| x\right)\right]. \end{aligned} $$

Remark 10:

$$ \tilde{R}_2(1) = \frac{(n!)^4}{(2n!)^2}.\sum_{k=1}^{\infty} \frac{((n+k)!)^4}{((2n+k)!)^2.k!^2}\left(\sum_{s=1}^{k} \frac{4}{n+1+s} - \frac{2}{2n+1+s} - \frac{2}{s}\right) $$

is equal to Beuker's integral

$$ -\int_0^1\int_0^1\int_0^1 \left(\frac{xyz(1-x)(1-y)(1-z)}{1-(1-xy)z}\right)^n . \frac{dxdydz}{1-(1-xy)z} $$

[Be2]!

Now by the previous study, the polynomial $A_3(x)$ is also a solution of this differential equation.

This Riemann scheme gives, as in the proof of Theorem 2, the hypergeometric polynomial

$$ A_3(x) = {}_4F_3\left(\begin{matrix} -n, -n, -n, -n \\ 2n+1, 2n+1, 1 \end{matrix} \middle| x\right). $$

This corresponds by an easy transformation of the Riemann scheme to Apéry's polynomial [Ap]

$$A_3(x) = \frac{x^n}{\binom{2n}{n}^2} {}_4F_3\left(\begin{matrix} -n, -n, n+1, n+1 \\ 1,1,1 \end{matrix} \middle| 1/x\right).$$

Let us recall the following definition [AAR].

Definition 5: A hypergeometric series

$${}_{m+1}F_m\left(\begin{matrix} a_0, a_1, \dots, a_m \\ b_1, \dots, b_m \end{matrix} \middle| x\right)$$

is called Saalschutzian if $x = 1$, if one of the parameter a_i (i.e. this series is a polynomial) is a negative integer, and if

$$1 + \sum_{i=0}^{m} a_i = \sum_{i=1}^{m} b_i.$$

$A_3(x)$ is "Saalschutzian".

Now, it can be shown that by substituting $x = 1$ in the left hand side one obtains

$$R_2(1) = 2A_3(1)\zeta(3) + A_0(1)$$

with

$$(4.5) \qquad a_n = A_3(1) = \sum_{l=0}^{l=n} \binom{n}{l}^2 \binom{n+l}{l}^2 = {}_4F_3\left(\begin{matrix} -n, -n, n+1, n+1 \\ 1,1,1 \end{matrix} \middle| 1\right),$$

precisely the approximations that Apéry used in his irrationality proof for $\zeta(3)$. (See also [Gu], [Ne2] and [Hu3].)

Recall that Apéry's original proof of the irrationality of $\zeta(3)$ made crucial use of the fact that the numbers a_n, $b_n := A_0(1)$ and $r_n := R_2(1)$ satisfy the second-order recurrence with polynomial coefficients,

$$(n+1)^3 a_{n+1} - ((n+1)^3 + n^3 + 4(2n+1)^3)a_n + n^3 a_{n-1} = 0.$$

Using the fact that $A_3(x)$ is a hypergeometric polynomial, we can also find such a relation by use of Zeilberger's algorithm of creative telescoping [Ze].

There exists another method to compute the polynomials $A_k(x)$ and the remainders $R_1(x)$ and $\tilde{R}_2(x)$ (see [Ne1]). If we put

$$Y(x) = \int_L x^t \Psi(t) dt,$$

the function $\Psi(t)$ satisfies the recurrence equation of the first order

$$(t+1) = \frac{(t-n)^4}{(t+1)^2(t-2n)^2}\Psi(t). \tag{4.6}$$

To solve (4.6) we must choose:

(1) The solution of this linear recurrence, which is a quotient of Γ functions modulo periodic functions of period 1.

(2) The path of integration L.

We choose, for example,

$$\Psi(t) = \frac{\pi^2\Gamma(t-n)^4}{\Gamma(t+1)^2\Gamma(t-2n-1)^2(\sin\pi t)^2}.$$

After easy simplifications of the gamma factors, putting

$$R(t) = \frac{((t-1)(t-2)\cdots(t-n))^2}{(t(t+1)(t+2)\cdots(t+n))^2},$$

a rational function, one finds

$$Y(x) = \int_L R(t)x^t.\frac{\pi^2}{(\sin\pi t)^2}dt.$$

We now expand $R(t)$ into the sum of partial functions

$$R(t) = \sum_{k=0}^{n}\frac{b_k}{(t+k)^2} + \sum_{k=0}^{n}\frac{a_k}{(t+k)};$$

we find

$$b_k = (t+k)^2R(t)|_{t=-k} = \binom{n+k}{k}^2\binom{n}{k}^2,$$

$$a_k = \frac{d}{dt}(t+k)^2R(t)|_{t=-k} = -2b_k\left(\sum_{j=1}^{n}\frac{1}{(j+k)} - \frac{1}{(k-j)}\right).$$

Thanks to residue calculus, we find for $\tilde{R}_2(x) = R_2(x) + \log xR_1(x)$

$$x^{2n+1}\left(\sum_{k=n+1}^{\infty}R'(k)x^{(k-n-1)}\right) + \left(\sum_{k=n+1}^{\infty}R(k)x^{(k-n-1)}\right)\log x.$$

See the end of the proof in [Gu].

5. Generalisation of Apéry's construction

Let us consider the deformation of the linear form

$$R_0(x) = A_0(x) + \sum_{k=1}^{m} A_k(x) Li_k(x),$$

$$R_0(x,s) = A_0(x,s)x^s + \sum_{k=1}^{m} A_k(x) Li_k(x,s),$$

with $A_0(x,0) = A_0(x)$, and

$$\tilde{R}_{p,k}(x) = \frac{1}{p!}\frac{\partial^p}{\partial s^p}(R_0(x,s))|_{s=0}. \tag{5.1}$$

Lemma 1: *For every positive integer p we have*

$$\mathcal{L}_{k,p}(x) = \frac{1}{p!}\frac{\partial^p}{\partial s^p}(Li_k(x,s))|_{s=0} = \sum_{j=0}^{p}(-1)^j \frac{\binom{k+j-1}{j}}{(p-j)!} Li_{k+j}(x).(\log x)^{p-j}. \tag{5.2}$$

Taylor's formula at $s = 0$ gives (formally)

$$R_0(x,s) = R_0(x) + \sum_{p=1}^{\infty}\left(\sum_{k=1}^{m} A_k(x)\mathcal{L}_{k,p}(x)\right) + R_{0,p}(x) \tag{5.3}$$

where

$$R_{0,p}(x) = \sum_{j=1}^{p} \frac{1}{j!(p-j)!}\frac{\partial^j}{\partial s^j} A_0(x,s)|_{s=0}.(\log x)^{p-j}.$$

Now if one puts

$$B_j(x) = \frac{\partial^j}{\partial s^j} A_0(x,s)|_{s=0}$$

and

$$\tilde{R}_{p,j}(x) = \sum_{k=1}^{m} A_k(x)(-1)^j \frac{\binom{k+j-1}{j}}{(p-j)!} Li_{k+j}(x) + B_j(x), \tag{5.4}$$

we obtain for $j = 0, 1, \ldots, p$

$$\tilde{R}_k(x) = \sum_{j=0}^{\infty} \tilde{R}_{p,j}(x)(\log x)^{p-j}. \tag{5.5}$$

Having disposed of these preliminary steps, we can proceed analogously to the proof of (2.2) for the p linear forms $\tilde{R}_{j,k}(x)$, $0 \le j \le p$.

The new local system is, for $1 \le j \le p$, generated by $\mathcal{L}_{p,k}(x)$ and is of rank $m+p+1$ over $\mathbb{C}(x)$.

The same proof as in Theorems 3 and 4 and the previous section gives in the first case (not well-poised) the following Riemann scheme related to (3.3):

$$R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ \sigma_0 & -n_0^{(1)} & 0 & \\ \vdots & & \vdots & \\ \sigma_0 & -n_0^{(p)} & p-1 & \\ \sigma_0 & -n_0 & p & \\ e_1 & -n_1 & p+1 & |x \\ e_2 & & & \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & & \\ e_{m-1} & & p+m & \\ 0 & -n_m & \sigma_1 & \end{pmatrix} \tag{5.6}$$

where for $1 \le k \le p$, $n_0^{(k)}$ denote the degree of the polynomials $\partial^k A(x,0)/\partial s^k$.

The remainders are given (within a multiplicative constant) by

$$R_0(x) = C(\sigma_0)x^{\sigma_0} \tag{5.7}$$
$$\times_{m+p+1}F_{m+p}\left(\begin{matrix}\sigma_0 - n_0^{(1)}, \dots, \sigma_0 - n_0^{(p)}, \sigma_0 - n_m, \sigma_0 - n_0, \dots, \sigma_0 - n_m \\ 1, \dots, 1, 1+\sigma_0 - e_1, \dots, 1+\sigma_0 - e_{m-1}\end{matrix} |x\right).$$

If we put

$$R_0(x) = x^{\sigma_0} \sum_{n=0}^{\infty} C_n x^n,$$

then the other remainders are given by the computation of

$$\frac{1}{k!}\frac{\partial^k}{\partial s^k}\left(x^{\sigma_0+s}\sum_{n=0}^{\infty} C_{(n+s)}x^n\right)|_{s=0}.$$

5.1 An example in the well-poised case. We also can find the same kind of formulae in the well-poised case. Putting $p=1$, $n_0 = n_1 = \dots = n_m = n_0^{(1)} = n$ and with the results of Theorem 4, one obtains

$$R_0(x) = C(\sigma_0)x^{\sigma_0} \tag{5.8}$$
$$\times_{m+3}F_{m+2}\left(\begin{matrix}2\sigma_0 - n, \sigma_0 - n, \sigma_0 - n \dots, \sigma_0 - n \\ 1+\sigma_0, \dots, 1+\sigma_0\end{matrix} |x\right),$$
$$R_0^1(x) = \frac{\partial}{\partial s}\left(x^{\sigma_0+s}\sum_{n=0}^{\infty} C_{(n+s)}x^n\right)|_{s=0};$$

then (within a multiplicative constant)

$$(5.9)\quad R_0^1(x) = R_0(x)\log x - [(m+1)A_m(x)Li_{m+1}(x) + \cdots + 2A_1(x)Li_2(x) + \partial A_0(x,0)/\partial s]$$

$$A_m(x) = {}_{m+2}F_{m+1}\left(\begin{matrix}\sigma_0 - n, , -n, -n, \ldots, -n \\ 1-\sigma_0, 1, 1, \ldots, 1\end{matrix} \middle| x\right).$$

If one chooses $m+1$ odd, we have $A_{m-1}(1) = \cdots = A_3(1) = A_1(1) = 0$. This gives the following linear form:

$$R_0^1(1) = (m+1)A_m(1)Li_{m+1}(1) + \cdots + 5A_5(1)Li_5(1) + A_0(1).$$

This formula is important for arithmetic applications [Zu1].

It is interesting to note that in the general case for $p \geq 3$, these formulae give the rational simultaneous approximations of

$$\zeta(m+p+1), \zeta(m+p-1), \ldots, \zeta(p+2), 1$$

for m and p odd integers [Zu1].

If $A_1(1) = 0$ then

$$(5.10)\qquad \tilde{R}_{p,j}(1) = \sum_{k=2}^{m} A_k(1)(-1)^j \frac{\binom{k+j-1}{j}}{(p-j)!}\zeta(k+j) + B_j(1).$$

If, for $1 \leq k \leq m$, d is a denominator of $A_k(1)$, $B_j(1)$, i.e. such that $dA_k(1) \in \mathbb{Z}$ resp $dB_j(1) \in \mathbb{Z}$ and $d\tilde{R}_{j,k}(1)$ is very small, then we obtain simultaneous rational approximations of

$$1, \zeta(j+2), \zeta(j+3), \ldots, \zeta(j+p).$$

6. Hermite–Padé approximation of the second kind for polylogarithmic functions

We try to find a polynomial $P_m(x)$ of degree mN such that the following conditions are satisfied:

$$(6.1)\quad \begin{aligned} R_1(1/x) &= P_m(x)Li_1(1/x) - Q_{1,N}(x) = O(1/x^{N+1}),\\ R_2(1/x) &= P_m(x)Li_2(1/x) - Q_{2,N}(x) = O(1/x^{N+1}),\\ &\vdots\\ R_m(1/x) &= P_m(x)Li_m(1/x) - Q_{m,N}(x) = O(1/x^{N+1}).\end{aligned}$$

For $1 \le k \le m$, $Q_{k,N}(x)$ are the polynomial parts of the expansion of

$$P_m(x) Li_k(1/x).$$

Indeed these conditions yield mN homogeneous linear relations for the $mN+1$ coefficients of $P_m(x)$.

Such a system always has a non trivial solution and so a polynomial $P_m(x) \not\equiv 0$ satisfying these conditions exists.

Now, using the construction of Lemma 1 and applying carefully the proof of Theorem 2, we find the following Riemann scheme:

$$R\begin{pmatrix} \underline{0} & \underline{\infty} & \underline{1} & \\ 0 & N+1 & 0 & \\ 0 & N+1 & 1 & \\ \vdots & \vdots & \vdots & |x \\ 0 & N+1 & m-1 & \\ 0 & -mN & 0 & \end{pmatrix}. \tag{6.2}$$

We obtain that

$$R_1(1/x), R_2(1/x), \dots, R_m(1/x), P_m(x)$$

are solutions of the hypergeometric differential equation

$$\theta^{m+1} - x(\theta - mN)(\theta + N + 1)^m = 0 \tag{6.3}$$

and (within a multiplicative constant) $P_m(x)$ is given by the following hypergeometric polynomial:

$$\begin{aligned} P_m(x) &= {}_{m+1}F_m\left(\begin{matrix} -mN, N+1, N+1, \dots, N+1 \\ 1, 1, \dots, 1, 1 \end{matrix} \Big| x\right) \\ &= \sum_{k=0}^{mN} \binom{mN}{k} \binom{N+k}{k}^m (-1)^k x^k. \end{aligned} \tag{6.4}$$

In the particular case $m = 1$ we obtain (within a multiplicatve constant) the Legendre polynomials.

One can use the following identity in $\mathbb{C}(x)(\theta)$ to prove that these polynomials (called 'Legendre type polynomials') are effectively generalisations of Legendre polynomials. Namely

$$D^k x^k = (\theta + 1) \cdots (\theta + k)$$

yields the relation

$$P_m(x) = \frac{1}{(N!)^m} \{(\theta+1)(\theta+2)\cdots(\theta+N)\}^m {}_{m+1}F_m\left(\begin{matrix} -mN, 1, 1, \dots, 1 \\ 1, 1, \dots, 1, 1 \end{matrix} \Big| x\right) \tag{6.5}$$

and the relation

$${}_1F_0\left(\begin{matrix}-mN\\ -\end{matrix}\,|x\right) = {}_{m+1}F_m\left(\begin{matrix}-mN,1,1,\ldots,1\\ 1,1,\ldots,1,1\end{matrix}\,|x\right)$$

gives for this hypergeometric polynomial the formula

$$P_m(x) = \frac{1}{(N!)^m}\{D^N x^N\}^m\{(1-x)^{mN}\}. \tag{6.6}$$

M. Hata ([Ha]) uses this formula to show that $P_m(x)$ is orthogonal on the interval $[0,1]$ to the m functions

$$x^j(\log x)^k, \quad 1 \le j \le N,\ 0 \le k < m.$$

The classical Legendre polynomials

$$p_N(x) = (-1)^N P_1(x)$$

satisfy the recurrence formula

$$(N+1)p_{N+1}(x) - (2N+1)(2x-1)p_N(x) + Np_{N-1}(x) = 0 \tag{6.7}$$

with

$$p_0(x) = 1, \quad p_1(x) = 2x - 1$$

and we recall that this gives the continued fraction expansion of $\log(1-1/x)$ at $x = \infty$.

This kind of recurrence was generalized by M. Hata [Ha], who proved that such polynomials satisfy a three-term recurrence of the form

$$P_{N+1}(x) = (a_N.x + b_N).P_N(x) + c_N.P_{N-1}(x).$$

If we know this recurrence explicitly, we can easily give a result of linear independence for the family

$$1, Li_1(x), Li_2(x), \ldots, Li_m(x)$$

by use of Poincaré's or Perron's theorems on asymptotics of linear recurrences ([MT] page 548).

Formula (6.7) gives the recurrence when $m = 1$. We can find it for $m = 2$ using the method of creative telescoping [Ze].

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